

Model Uncertainty Stochastic Mean-Field Control

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6 January 2017

MSC(2010): 60H05, 60H20, 60J75, 93E20, 91G80, 91B70.

Keywords: Mean-field stochastic differential equation; measure-valued optimal control; model uncertainty; stochastic differential game; stochastic maximum principle; operator-valued backward stochastic differential equation; optimal consumption of a mean-field cash flow under model uncertainty.

Abstract

We consider the problem of optimal control of a mean-field stochastic differential equation (SDE) under model uncertainty. The model uncertainty is represented by ambiguity about the law $\mathcal{L}(X(t))$ of the state $X(t)$ at time t . For example, it could be the law $\mathcal{L}_{\mathbb{P}}(X(t))$ of $X(t)$ with respect to the given, underlying probability measure \mathbb{P} . This is the classical case when there is no model uncertainty. But it could also be the law $\mathcal{L}_{\mathbb{Q}}(X(t))$ with respect to some other probability measure \mathbb{Q} or, more generally, any random measure $\mu(t)$ on \mathbb{R} with total mass 1.

We represent this model uncertainty control problem as a *stochastic differential game* of a mean-field related type SDE with two players. The control of one of the players, representing the uncertainty of the law of the state, is a measure-valued stochastic process $\mu(t)$ and the control of the other player is a classical real-valued stochastic process $u(t)$. This optimal control problem with respect to random probability processes $\mu(t)$ in a non-Markovian setting is a new type of stochastic control problems that has not been studied before. By introducing operator-valued backward stochastic differential equations (BSDE), we obtain a sufficient and a necessary maximum principle for Nash equilibria for such games in the general nonzero-sum case, and for saddle points

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²This research was carried out with support of the Norwegian Research Council, within the research project Challenges in Stochastic Control, Information and Applications (STOCONINF), project number 250768/F20.

³This work was presented by one of us (Nacira Agram) at the conference "Model Uncertainty & Robust Finance" at the University of Milan, Italy, 9-11 November 2016. We want to thank the organizers for the financial support and the participants for their helpful comments.

in zero-sum games.

As an application we find an explicit solution of the problem of optimal consumption under model uncertainty of a cash flow described by a mean-field related type SDE.

1 Introduction

There are many ways of introducing model uncertainty. For example, in recent works of Øksendal et al [13]-[12] and Menoukeu Pamen [9], the underlying probability measure is not given a priori and there can be a family of possible probability measures to choose from.

The aim of this paper is to study stochastic optimal control under model uncertainty of a mean-field related type SDE driven by Brownian motion and an independent Poisson random measure. The model uncertainty is represented by ambiguity about the law $\mathcal{L}(X(t))$ of the state $X(t)$ at time t . For example, it could be the law $\mathcal{L}_{\mathbb{P}}(X(t))$ of $X(t)$ with respect to the given, underlying probability measure \mathbb{P} . This is the classical case when there is no model uncertainty. But it could also be the law $\mathcal{L}_{\mathbb{Q}}(X(t))$ with respect to some other probability measure \mathbb{Q} or, more generally, any random measure $\mu(t)$ on \mathbb{R} with total mass 1.

We represent this model uncertainty control problem as a *stochastic differential game* of a mean-field related type SDE with two players. The control of one of the players, representing the uncertainty of the law of the state, is a measure-valued stochastic process $\mu(t)$, and the control of the other player is a classical real-valued stochastic process $u(t)$. We penalize $\mu(t)$ for being far away from the law $\mathcal{L}_{\mathbb{P}}(X(t))$ with respect to the original probability measure \mathbb{P} . This leads to a new type of mean-field stochastic control problems in which the control is random measure-valued stochastic process $\mu(t)$ on \mathbb{R} . To the best of our knowledge this type of problem has not been studied before. By introducing a new type of adjoint, operator-valued BSDEs, we obtain sufficient and necessary maximum principles for Nash equilibria for such games in the general nonzero-sum case, and saddle points for zero-sum games.

As an application we find an explicit solution of the problem of optimal consumption under model uncertainty of a cash flow described by a mean-field related type SDE.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ generated by a one-dimensional Brownian motion B and an independent Poisson random measure $N(dt, d\zeta)$. Let $\nu(d\zeta)dt$ denote the Lévy measure of N , and let $\tilde{N}(dt, d\zeta)$ denote the compensated Poisson random measure $N(dt, d\zeta) - \nu(d\zeta)dt$. The probability \mathbb{P} is a reference probability measure. We introduce two smaller filtrations $\mathbb{G}^{(i)} = (\mathcal{G}_t^{(i)})_{t \geq 0}$ such that $\mathcal{G}_t^{(i)} \subseteq \mathcal{F}_t$, for $i = 1, 2$ and for all $t \geq 0$. These filtrations represent the information available to player number i at time t .

We first recall some concepts, spaces and definitions which will be used on the sequel.

2.1 Some basic concepts from Banach space theory

Let \mathcal{X}, \mathcal{Y} be two Banach spaces with norms $\|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{Y}}$, respectively, and let $F : \mathcal{X} \rightarrow \mathcal{Y}$.

- We say that F has a directional derivative (or Gâteaux derivative) at $v \in \mathcal{X}$ in the direction $w \in \mathcal{X}$ if

$$D_w F(v) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (F(v + \varepsilon w) - F(v))$$

exists.

- We say that F is Fréchet differentiable at $v \in \mathcal{X}$ if there exists a continuous linear map $A : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\lim_{\substack{h \rightarrow 0 \\ h \in \mathcal{X}}} \frac{1}{\|h\|_{\mathcal{X}}} \|F(v + h) - F(v) - A(h)\|_{\mathcal{Y}} = 0.$$

In this case we call A the *gradient* (or Fréchet derivative) of F at v and we write

$$A = \nabla_v F.$$

- If F is Fréchet differentiable at v with Fréchet derivative $\nabla_v F$, then F has a directional derivative in all directions $w \in \mathcal{X}$ and

$$D_w F(v) := \langle \nabla_v F, w \rangle = \nabla_v F(w) = \nabla_v F w.$$

In particular, note that if F is a linear operator, then $\nabla_v F = F$ for all v .

2.2 t-absolute continuity and t-derivative of the law process

Suppose that $X(t) = X_t$ is an Itô-Lévy process of the form

$$\begin{cases} dX_t = \alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t, \zeta) \tilde{N}(dt, d\zeta); & t \in [0, T], \\ X_0 = x \in \mathbb{R}, \end{cases} \quad (2.1) \quad \{\text{eq2.1}\}$$

where α, β and γ are bounded predictable processes.

Let $\varphi \in C^2$. Then under appropriate conditions on the coefficients, we get by the Itô formula

$$\mathbb{E}[\varphi(X_{t+h})] - \mathbb{E}[\varphi(X_t)] = \mathbb{E}[\int_t^{t+h} A\varphi(X_s)ds], \quad (2.2) \quad \{\text{eq2.3}\}$$

where

$$\begin{aligned} A\varphi(X_s) &= \alpha(s)\varphi'(X_s) + \frac{1}{2}\beta^2(s)\varphi''(X_s) \\ &\quad + \int_{\mathbb{R}} \{\varphi(X_s + \gamma(s, \zeta)) - \varphi(X_s) - \varphi'(X_s)\gamma(s, \zeta)\} \nu(d\zeta). \end{aligned}$$

In particular, if

$$\varphi(x) = \varphi_y(x) := \exp(ixy); \quad y \in \mathbb{R},$$

then

$$\begin{aligned} A\varphi_y(X_s) &= \left(iy\alpha(s) - \frac{1}{2}\beta^2(s)y^2 \right. \\ &\quad \left. + \int_{\mathbb{R}} \{ \exp(i\gamma(s, \zeta)y) - 1 - iy\gamma(s, \zeta) \} \nu(d\zeta) \right) \varphi_y(X_s), \end{aligned}$$

for all $y \in \mathbb{R}$.

We now introduce the following Banach space:

Definition 2.1 Define \mathcal{M} to be the Banach space of all random probability measures $\mu = \mu(\omega)$ on \mathbb{R} with the norm $\|\cdot\|_{\mathcal{M}}$ defined by

$$\|\mu\|_{\mathcal{M}}^2 := \mathbb{E}[\int_{\mathbb{R}} |\hat{\mu}(y)|^2 e^{-y^2} dy], \quad (2.3) \quad \{2.6\}$$

where

$$\hat{\mu}(y) := \int_{\mathbb{R}} e^{ixy} d\mu(x) \quad (2.4) \quad \{2.7\}$$

is the Fourier transform of the measure μ .

Let \mathcal{M}_0 be the set of deterministic elements of \mathcal{M} .

Definition 2.2 (Law process)

From now on we use the notation

$$M_t := \mathcal{L}(X_t); \quad 0 \leq t \leq T \quad (2.5)$$

for the law process $\mathcal{L}(X_t)$ of X_t with respect to \mathbb{P} .

Lemma 2.3 The map $t \mapsto M_t : [0, T] \rightarrow \mathcal{M}_0$ is absolutely continuous.

Proof. Let $0 \leq t < t+h \leq T$. Then by (2.3) and (2.4), we get

$$\begin{aligned} \|M_{t+h} - M_t\|_{\mathcal{M}}^2 &= \int_{\mathbb{R}} |\hat{M}_{t+h}(y) - \hat{M}_t(y)|^2 e^{-y^2} dy \\ &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{ixy} d\mathcal{L}(X_{t+h})(x) - \int_{\mathbb{R}} e^{ixy} d\mathcal{L}(X_t)(x) \right|^2 e^{-y^2} dy \\ &= \int_{\mathbb{R}} |\mathbb{E}[\varphi_y(X_{t+h})] - \mathbb{E}[\varphi_y(X_t)]|^2 e^{-y^2} dy. \end{aligned} \quad (2.6) \quad \{2.8\}$$

The last equality holds by using that for any bounded function ψ we have

$$\mathbb{E}[\psi(X)] = \int_{\mathbb{R}} \psi(x) d\mathcal{L}(X)(x).$$

By (2.2), we obtain

$$\begin{aligned} \|M_{t+h} - M_t\|_{\mathcal{M}}^2 &= \int_{\mathbb{R}} \left| \mathbb{E} \left[\int_t^{t+h} A\varphi_y(X(s)) ds \right] \right|^2 e^{-y^2} dy \\ &\leq \int_{\mathbb{R}} \left(\int_t^{t+h} \mathbb{E}[|A\varphi_y(X_s)|] ds \right)^2 e^{-y^2} dy \leq C h^2, \end{aligned}$$

for some constant C which does not depend on t and h .

We have proved that for different t and $t+h$, $\|M_{t+h} - M_t\|_{\mathcal{M}} \leq C h$ and it is easy to see that this holds for every finite disjoint partition of the interval $[0, T]$. \square

From the lemma above we conclude the following:

Lemma 2.4 *If X_t is an Itô-Lévy process as in (2.1), then the derivative $M'_s := \frac{d}{ds}M_s$ exists in \mathcal{M}_0 , and we have*

$$M_t = M_0 + \int_0^t M'_s ds; \quad t \geq 0.$$

In the following we will apply this to the solutions $X(t)$ of the mean-field related type SDEs we consider below.

2.3 Spaces

Throughout this work, we will use the following spaces:

- \mathcal{S}^2 is the set of \mathbb{R} -valued \mathbb{F} -adapted càdlàg processes $(X(t))_{t \in [0, T]}$ such that

$$\|X\|_{\mathcal{S}^2}^2 := \mathbb{E}[\sup_{t \in [0, T]} |X(t)|^2] < \infty.$$

- L^2 is the set of \mathbb{R} -valued \mathcal{F}_t -adapted processes $(Q(t))_{t \in [0, T]}$ such that

$$\|Q\|_{L^2}^2 := \mathbb{E}[\int_0^T |Q(t)|^2 dt] < \infty.$$

- $L^2(\mathcal{F}_t)$ is the set of \mathbb{R} -valued square integrable \mathcal{F}_t -measurable random variables.
- L_ν^2 is the set of Borel functions $R : \mathbb{R}_0 := \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ such that

$$\|R\|_{L_\nu^2}^2 := \mathbb{E}[\int_{\mathbb{R}_0} |R(\zeta)|^2 \nu d\zeta] < \infty.$$

- We say that $\mu(t)$ is *adapted* to a given filtration \mathbb{H} if $\mu(t)(V)$ is \mathbb{H} -adapted for all Borel sets $V \subseteq \mathbb{R}$. Let $\mathbb{M}_{\mathbb{G}} = \mathbb{M}_{\mathbb{G}(1)}$ be a given set of \mathcal{M} -valued, $\mathbb{G}^{(1)} = (\mathcal{G}_t^{(1)})_{t \geq 0}$ -adapted, stochastic processes $\mu(t)$. We call $\mathbb{M}_{\mathbb{G}}$ the set of *admissible measure-valued control processes* $\mu(\cdot)$.
- Let $\mathcal{A}_{\mathbb{G}} = \mathcal{A}_{\mathbb{G}(2)}$ be a given set of real-valued, $\mathbb{G}^{(2)} = (\mathcal{G}_t^{(2)})_{t \geq 0}$ -adapted, stochastic processes u required to have values in a given convex subset \mathcal{U} of \mathbb{R} . We call $\mathcal{A}_{\mathbb{G}}$ the set of *admissible real-valued control processes* $u(\cdot)$.
- \mathcal{R} is the set of measurable functions $k : \mathbb{R}_0 \rightarrow \mathbb{R}$.
- $C_a([0, T], \mathcal{M}_0)$ denotes the set of absolutely continuous functions $m : [0, T] \rightarrow \mathcal{M}_0$.
- \mathcal{K} is the set of bounded linear functionals $r : \mathcal{M}_0 \rightarrow \mathbb{R}$.

3 The model uncertainty stochastic optimal control problem

In this paper, we are interested in systems governed by controlled mean-field related type SDE $X^{\mu,u}(t) = X(t) \in \mathcal{S}^2$ on the form

$$\begin{cases} dX(t) &= b(t, X(t), \mu(t), u(t)) dt + \sigma(t, X(t), \mu(t), u(t)) dB(t) \\ &\quad + \int_{\mathbb{R}_0} \gamma(t, X(t), \mu(t), u(t), \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ X(0) &= x \in \mathbb{R}. \end{cases} \quad (3.1) \quad \{\text{sde}\}$$

The functions

$$\begin{aligned} b(t, x, \mu, u) &= b(t, x, \mu, u, \omega) : [0, T] \times \mathbb{R} \times \mathcal{M} \times \mathcal{U} \times \Omega \rightarrow \mathbb{R}, \\ \sigma(t, x, \mu, u) &= \sigma(t, x, \mu, u, \omega) : [0, T] \times \mathbb{R} \times \mathcal{M} \times \mathcal{U} \times \Omega \rightarrow \mathbb{R}, \\ \gamma(t, x, \mu, u, \zeta) &= \gamma(t, x, \mu, u, \zeta, \omega) : [0, T] \times \mathbb{R} \times \mathcal{M} \times \mathcal{U} \times \mathbb{R}_0 \times \Omega \rightarrow \mathbb{R}, \end{aligned}$$

are supposed to be Lipschitz on $x \in \mathbb{R}$, uniformly w.r.t t and ω for given $u \in \mathcal{U}$ and $\mu \in \mathcal{M}$. Then by e.g. Theorem 1.19 in [10], we have existence and uniqueness of the solution of $X(t)$.

Let us consider a performance functional of the form

$$J(\mu, u) = \mathbb{E}[g(X(T), M(T)) + \int_0^T \ell(s, X(s), M(s), \mu(s), u(s)) ds], \quad (3.2) \quad \{\text{performance}\}$$

where $\ell(t, x, m, \mu, u) = \ell(t, x, m, \mu, u, \omega) : [0, T] \times \mathbb{R} \times \mathcal{M} \times \mathcal{M} \times \mathcal{U} \times \Omega \rightarrow \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{M} \times \Omega \rightarrow \mathbb{R}$ are given functions.

For fixed x, m, μ, u we assume that $\ell(s, \cdot)$ is \mathcal{F}_s -measurable and $g(\cdot, m)$ is \mathcal{F}_T -measurable. We also assume the following integrability condition

$$\mathbb{E}[|g(X(T), M(T))| + \int_0^T |\ell(s, X(s), M(s), \mu(s), u(s))| ds] < \infty,$$

for all $\mu \in \mathbb{M}_{\mathbb{G}}$ and $u \in \mathcal{A}_{\mathbb{G}}$.

Note that the system (3.1) and the performance (3.2) are not Markovian. Therefore dynamic programming cannot be used to study the corresponding optimal control problem. In the next section we study a stochastic differential game of two players, where one of the players is solving an optimal measure-valued control problem of the type described above, while the other player is solving a classical real-valued stochastic control problem. To the best of our knowledge this type of stochastic differential game has not been studied before.

The interpretation of the system (3.1) is that it is a perturbation (model uncertainty) of the mean-field equation

$$\begin{cases} dX(t) = \sigma(t, X(t), \mathcal{L}(X(t)), u(t)) dB(t) + b(t, X(t), \mathcal{L}(X(t)), u(t)) dt \\ \quad + \int_{\mathbb{R}_0} \gamma(t, X(t), \mathcal{L}(X(t)), u(t), \zeta) \tilde{N}(dt, d\zeta), t \in [0, T], \\ X(0) = x \in \mathbb{R}. \end{cases}$$

For example, we could have $\mu(t) = \mathcal{L}_{\mathbb{Q}}(X(t))$ for some probability measure $\mathbb{Q} \neq \mathbb{P}$ and in that case expectations w.r.t \mathbb{P} should be changed to expectations w.r.t \mathbb{Q} as follows

$$J(\mu, u) = \mathbb{E}_{\mathbb{Q}} \left[g(X(T)) + \int_0^T \ell(s, X(s), \mu(s), u(s)) ds \right].$$

If $\mathbb{Q} \ll \mathbb{P}$ then this is equivalent to

$$J(\mu, u) = \mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}}(g(X(T)) + \int_0^T \ell(s, X(s), \mu(s), u(s)) ds) \right],$$

where we have denoted $\mathbb{E}_{\mathbb{P}}$ by just \mathbb{E} and $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is the Radon-Nikodym derivative of \mathbb{Q} w.r.t. \mathbb{P} .

The case when J does not depend on μ with this change of probability measure was studied by many authors see for example [7],[11],[12],[13],[9]. The case when the coefficients depend on the law of state has been studied by many authors for example [8], [4], [6], [3], [2] and it was recently extend to the law of the anticipated solution by [1].

4 Nonzero-sum games

We now proceed to a nonzero-sum maximum principle.
The cost functionals are assumed to be on the form

$$J_i(\mu, u) = \mathbb{E} \left[g_i(X(T), M(T)) + \int_0^T \ell_i(s, X(s), M(s), \mu(s), u(s)) ds \right]; \text{ for } i = 1, 2, \quad (4.1) \quad \{\text{perf}\}$$

where $M(s) := \mathcal{L}(X(s))$ and the functions

$$\begin{aligned} \ell_i(t, x, m, \mu, u) &= \ell_i(t, x, m, \mu, u, \omega) : [0, T] \times \mathbb{R} \times \mathcal{M} \times \mathcal{M} \times \mathcal{U} \times \Omega \rightarrow \mathbb{R}, \\ g_i(x, m) &= g_i(x, m, \omega) : \mathbb{R} \times \mathcal{M} \times \Omega \rightarrow \mathbb{R}, \end{aligned}$$

are continuously differentiable w.r.t x, u and admit Fréchet derivatives w.r.t m and μ .

Problem 4.1 *We consider the general nonzero-sum stochastic game to find $(\mu^*, u^*) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ such that*

$$\begin{aligned} J_1(\mu, u^*) &\leq J_1(\mu^*, u^*), \text{ for all } \mu \in \mathbb{M}_{\mathbb{G}}, \\ J_2(\mu^*, u) &\leq J_2(\mu^*, u^*), \text{ for all } u \in \mathcal{A}_{\mathbb{G}}. \end{aligned}$$

The pair (μ^, u^*) is called a Nash equilibrium.*

Definition 4.2 (The Hamiltonian) *For $i = 1, 2$ we define the Hamiltonian*

$$H_i : [0, T] \times \mathbb{R} \times C_a([0, T], \mathcal{M}_0) \times \mathcal{M} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \mathcal{K} \rightarrow \mathbb{R}$$

by

$$\begin{aligned} H_i(t, x, m, \mu, u, p_i^0, q_i^0, r_i^0(\cdot), p_i^1) &= \ell_i(t, x, m, \mu, u) + p_i^0 b(t, x, \mu, u) + q_i^0 \sigma(t, x, \mu, u) \\ &+ \int_{\mathbb{R}_0} r_i^0(\zeta) \gamma(t, x, \mu, u, \zeta) \nu(d\zeta) + \langle p_i^1, m' \rangle, \end{aligned} \quad (4.2) \quad \{\text{ham1}\}$$

where $m' = \frac{d}{dt}m$.

We assume that H_i is continuously differentiable w.r.t. x, u and admits Fréchet derivatives w.r.t. m and μ .

For $u \in \mathcal{A}_G, \mu \in \mathbb{M}_G$ with corresponding solution $X = X^{\mu, u}$, define $p_i = p_i^{\mu, u}, q_i = q_i^{\mu, u}$ and $r_i = r_i^{\mu, u}$ by the following set of adjoint equations:

- The real-valued BSDE in the unknown $(p_i^0, q_i^0, r_i^0) \in \mathcal{S}^2 \times L^2 \times L_\nu^2$ given by

$$\begin{cases} dp_i^0(t) &= -\frac{\partial H_i}{\partial x}(t)dt + q_i^0(t)dB(t) + \int_{\mathbb{R}_0} r_i^0(t, \zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ p_i^0(T) &= \frac{\partial g_i}{\partial x}(X(T), M(T)), \end{cases} \quad (4.3) \quad \{\text{eqp0}\}$$

- and the operator-valued BSDE in the unknown $(p_i^1, q_i^1, r_i^1) \in \mathcal{S}^2 \times L^2 \times L_\nu^2$ given by

$$\begin{cases} dp_i^1(t) &= -\nabla_m H_i(t)dt + q_i^1(t)dB(t) + \int_{\mathbb{R}_0} r_i^1(t, \zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ p_i^1(T) &= \nabla_m g_i(X(T), M(T)), \end{cases} \quad (4.4) \quad \{\text{eqp1}\}$$

where $H_i(t) = (X(t), M(t), \mu(t), u(t), p_i(t), q_i(t), r_i(t, \cdot), p_i^1(t))$ etc.

We remark that these two BSDEs (4.3) and (4.4) are linear, so whenever knowing the Hamiltonian H_i and the function g_i , we can get a solution explicitly. To remind the reader of this solution formula, let us consider the solution $(P, Q, R) \in \mathcal{S}^2 \times L^2 \times L_\nu^2$ of the linear BSDE

$$\begin{cases} dP(t) &= -[\varphi(t) + \alpha(t)P(t) + \beta(t)Q(t) + \int_{\mathbb{R}_0} \phi(t, \zeta)R(t, \zeta)\nu(d\zeta)]dt \\ &\quad + Q(t)dB(t) + \int_{\mathbb{R}_0} R(t, \zeta)\tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ Y(T) &= \theta \in L^2(\mathcal{F}_T). \end{cases} \quad (4.5) \quad \{\text{lbsde}\}$$

Here φ, α, β and ϕ are bounded predictable processes with ϕ is assumed to be an \mathbb{R} -valued process defined on $[0, T] \times \mathbb{R}_0 \times \Omega$. Then it is well-known (see e.g. Theorem 1.7 in Øksendal and Sulem [11]) that the solution of equation (4.5) is given by

$$P(t) = \mathbb{E} \left[\theta \frac{\Gamma(T)}{\Gamma(t)} + \int_t^T \frac{\Gamma(s)}{\Gamma(t)} \varphi(s) \middle| \mathcal{F}_t \right]; \quad t \in [0, T], \quad (4.6) \quad \{\text{solution}\}$$

where $\Gamma(t) \in \mathcal{S}^2$ is the solution of the SDE

$$\begin{cases} d\Gamma(t) &= \Gamma(t^-) \left[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} R(t, \zeta)\tilde{N}(dt, d\zeta) \right]; \quad t \in [0, T], \\ \Gamma(0) &= 1. \end{cases} \quad (4.7) \quad \{\text{gama}\}$$

For notational convenience, we will employ the following short hand notations

$$\begin{aligned} \hat{H}_1(t) &= H_1(t, \hat{X}(t), \hat{M}(t), \hat{\mu}(t), \hat{u}(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}_1(t, \cdot), \hat{p}_1^1(t)), \\ \check{H}_1(t) &= H_1(t, \hat{X}(t), \hat{M}(t), \mu(t), \hat{u}(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}_1(t, \cdot), \hat{p}_1^1(t)), \\ \bar{H}_2(t) &= H_2(t, \hat{X}(t), \hat{M}(t), \hat{\mu}(t), \hat{u}(t), \hat{p}_2(t), \hat{q}_2(t), \hat{r}_2(t, \cdot), \hat{p}_2^1(t)), \\ \check{H}_2(t) &= H_2(t, \hat{X}(t), \hat{M}(t), \hat{\mu}(t), u(t), \hat{p}_2(t), \hat{q}_2(t), \hat{r}_2(t, \cdot), \hat{p}_2^1(t)). \end{aligned}$$

Similar notation is used for the derivatives of H and etc.

We now state a sufficient theorem for the nonzero-sum games.

Theorem 4.3 (Sufficient nonzero-sum maximum principle) *Let $(\hat{\mu}, \hat{u}) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ with corresponding solutions \hat{X} , (p_i^0, q_i^0, r_i^0) and (p_i^1, q_i^1, r_i^1) of the forward and backward stochastic differential equations (3.1), (4.3) and (4.4) respectively. Suppose that*

1. (Concavity) *The functions*

$$\begin{aligned} (x, m, \mu) &\mapsto H_1(t) \\ (x, m, u) &\mapsto H_2(t) \\ (x, m) &\mapsto g_i(x, m), \text{ for } i = 1, 2, \end{aligned}$$

are concave \mathbb{P} .a.s for each $t \in [0, T]$.

2. (Maximum conditions)

$$\mathbb{E} \left[\hat{H}_1(t) \middle| \mathcal{G}_t^{(1)} \right] = \operatorname{ess\,sup}_{\mu \in \mathbb{M}_{\mathbb{G}}} \mathbb{E} \left[\check{H}_1(t) \middle| \mathcal{G}_t^{(1)} \right], \quad (4.8) \quad \{\max Q\}$$

and

$$\mathbb{E} \left[\bar{H}_2(t) \middle| \mathcal{G}_t^{(2)} \right] = \operatorname{ess\,sup}_{u \in \mathcal{A}_{\mathbb{G}}} \mathbb{E} \left[\check{H}_2(t) \middle| \mathcal{G}_t^{(2)} \right],$$

\mathbb{P} .a.s for each $t \in [0, T]$.

Then $(\hat{\mu}, \hat{u})$ is a Nash equilibrium for our problem.

Proof. Let us first prove that $J_1(\mu, \hat{u}) \leq J_1(\hat{\mu}, \hat{u})$.

By the definition of the cost functional (4.1) we have for fixed $\hat{u} \in \mathcal{A}_{\mathbb{G}}$ and arbitrary $\mu \in \mathbb{M}_{\mathbb{G}}$

$$J_1(\mu, \hat{u}) - J_1(\hat{\mu}, \hat{u}) = I_1 + I_2, \quad (4.9) \quad \{j\}$$

where

$$\begin{aligned} I_1 &= \mathbb{E} \left[\int_0^T \left\{ \check{\ell}_1(t) - \hat{\ell}_1(t) \right\} dt \right], \\ I_2 &= \mathbb{E} \left[\check{g}_1(X(T), M(T)) - \hat{g}_1(\hat{X}(T), \hat{M}(T)) \right]. \end{aligned}$$

By the definition of the Hamiltonian (4.2) we have

$$I_1 = \mathbb{E} \left[\int_0^T \check{H}_1(t) - \hat{H}_1(t) - \hat{p}_1^0(t) \tilde{b}(t) - \hat{q}_1^0(t) \tilde{\sigma}(t) - \int_{\mathbb{R}_0} \hat{r}_1^0(t, \zeta) \tilde{\gamma}(t, \zeta) \nu(d\zeta) - \hat{p}_1^1(t) \tilde{M}'(t) dt \right], \quad (4.10) \quad \{i1\}$$

where $\tilde{b}(t) = \check{b}(t) - \hat{b}(t)$ etc., and

$$\tilde{M}'(t) = \frac{d\tilde{M}(t)}{dt} = \frac{d}{dt} \mathcal{L}(\tilde{X}(t)).$$

By the concavity of g_1 and the terminal values of the BSDEs (4.3), (4.4), we have

$$I_2 \leq \mathbb{E} \left[\frac{\partial g_1}{\partial x}(T) \tilde{X}(T) + \nabla_m g_1(T) \tilde{M}(T) \right] = \mathbb{E} \left[\hat{p}_1^0(T) \tilde{X}(T) + \hat{p}_1^1(T) \tilde{M}(T) \right].$$

Applying the Itô formula to $\hat{p}_1^0 \tilde{X}$ and $\hat{p}_1^1 \tilde{M}$, we get

$$\begin{aligned}
I_2 &\leq \mathbb{E}[\hat{p}_1^0(T) \tilde{X}(T) + \hat{p}_1^1(T) \tilde{M}(T)] \\
&= \mathbb{E} \left[\int_0^T \hat{p}_1^0(t) d\tilde{X}(t) + \int_0^T \tilde{X}(t) d\hat{p}_1^0(t) + \int_0^T \hat{q}_1^0(t) \tilde{\sigma}(t) dt + \int_0^T \int_{\mathbb{R}_0} \hat{r}_1^0(t, \zeta) \tilde{\gamma}(t, \zeta) \nu(d\zeta) dt \right] \\
&+ \mathbb{E} \left[\int_0^T \hat{p}_1^1(t) d\tilde{M}(t) + \int_0^T \tilde{M}(t) d\hat{p}_1^1(t) \right] \\
&= \mathbb{E} \left[\int_0^T \hat{p}_1^0(t) \tilde{b}(t) dt - \int_0^T \frac{\partial \hat{H}_1}{\partial x}(t) \tilde{X}(t) dt + \int_0^T \hat{q}_1^0(t) \tilde{\sigma}(t) dt \right. \\
&\left. + \int_0^T \int_{\mathbb{R}_0} \hat{r}_1^0(t, \zeta) \tilde{\gamma}(t, \zeta) \nu(d\zeta) dt + \int_0^T \hat{p}_1^1(t) \tilde{M}'(t) dt - \int_0^T \nabla_m \hat{H}_1(t) \tilde{M}(t) dt \right], \tag{4.11} \quad \{\text{I2}\}
\end{aligned}$$

where we have used that the $dB(t)$ and $\tilde{N}(dt, d\zeta)$ integrals with the necessary integrability property are martingales and then have mean zero. Substituting (4.10) and (4.11) in (4.9), yields

$$J_1(\mu, \hat{u}) - J_1(\hat{\mu}, \hat{u}) \leq \mathbb{E} \left[\int_0^T \{ \tilde{H}_1(t) - \hat{H}_1(t) - \frac{\partial \hat{H}_1}{\partial x}(t) \tilde{X}(t) - \nabla_m \hat{H}_1 \tilde{M}(t) \} dt \right].$$

By the concavity of H_1 and the fact that the process μ is $\mathcal{G}_t^{(1)}$ -adapted, we obtain

$$\begin{aligned}
J_1(\mu, \hat{u}) - J_1(\hat{\mu}, \hat{u}) &\leq \mathbb{E} \left[\int_0^T \frac{\partial \hat{H}_1}{\partial \mu}(t) (\mu(t) - \hat{\mu}(t)) dt \right] \\
&= \mathbb{E} \left[\int_0^T \mathbb{E} \left(\frac{\partial \hat{H}_1}{\partial \mu}(t) (\mu(t) - \hat{\mu}(t)) \middle| \mathcal{G}_t^{(1)} \right) dt \right] \\
&= \mathbb{E} \left[\int_0^T \mathbb{E} \left(\frac{\partial \hat{H}_1}{\partial \mu}(t) \middle| \mathcal{G}_t^{(1)} \right) (\mu(t) - \hat{\mu}(t)) dt \right] \\
&\leq 0,
\end{aligned}$$

where $\frac{\partial \hat{H}_1}{\partial \mu} = \nabla_\mu \hat{H}_1$. The last equality holds because of the maximum condition of \hat{H}_1 at $\mu = \hat{\mu}$.

Similar considerations apply to prove that $J_2(\hat{\mu}, u) \leq J_2(\hat{\mu}, \hat{u})$. For the sake of completeness, we give details in the Appendix. \square

We now state and prove a necessary version of the maximum principle. We assume the following:

- Whenever $\mu \in \mathbb{M}_{\mathbb{G}}$ ($u \in \mathcal{A}_{\mathbb{G}}$) and $\eta \in \mathbb{M}_{\mathbb{G}}$ ($\pi \in \mathcal{A}_{\mathbb{G}}$) is bounded, there exists $\epsilon > 0$ such that

$$\mu + \lambda \eta \in \mathbb{M}_{\mathbb{G}} \quad (\eta + \lambda \pi \in \mathcal{A}_{\mathbb{G}}), \text{ for each } \lambda \in [-\epsilon, \epsilon].$$

- For each $t_0 \in [0, T]$ and each bounded $\mathcal{G}_{t_0}^{(1)}$ -measurable random measure α_1 and $\mathcal{G}_{t_0}^{(2)}$ -measurable random variable α_2 , the process

$$\eta(t) = \alpha_1 1_{[t_0, T]}(t) \tag{4.12} \quad \{\text{eta}\}$$

belongs to $\mathbb{M}_{\mathbb{G}}$ and the process

$$\pi(t) = \alpha_2 1_{[t_0, T]}(t)$$

belongs to $\mathcal{A}_{\mathbb{G}}$.

- The *derivative* of the state $X(t)$, defined by (3.1) is

$$Z(t) := \left. \frac{d}{d\lambda} X^{\mu+\lambda\eta} \right|_{\lambda=0} =: DX(t)$$

exists, and is given by

$$\begin{cases} dZ(t) &= \left[\frac{\partial b}{\partial x}(t) Z(t) + \frac{\partial b}{\partial \mu}(t) \eta(t) \right] dt + \left[\frac{\partial \sigma}{\partial x}(t) Z(t) + \frac{\partial \sigma}{\partial \mu}(t) \eta(t) \right] dB(t) \\ &+ \int_{\mathbb{R}_0} \left[\frac{\partial \gamma}{\partial x}(t, \zeta) Z(t) + \frac{\partial \gamma}{\partial \mu}(t, \zeta) \eta(t) \right] \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ Z(0) &= 0. \end{cases} \quad (4.13) \quad \{\text{dervz}\}$$

We remark that this derivative process is a linear SDE, then by assuming that b , σ and γ admit bounded partial derivatives w.r.t x and μ , there is a unique solution $Z(t) \in \mathcal{S}^2$ of (4.13).

We want to prove that $Z(t)$ is exactly the L^2 -derivative of $X^{\mu+\lambda\eta}(t)$ w.r.t λ at $\lambda = 0$. More precisely, we want to prove the following.

Lemma 4.4

$$\mathbb{E} \left[\int_0^T \left(\frac{X^{\mu+\lambda\eta}(s) - X^\mu(s)}{\lambda} - Z(s) \right)^2 ds \right] \rightarrow 0 \text{ as } \lambda \rightarrow 0. \quad (4.14) \quad \{\text{vraider}\}$$

Proof. For notational convenience, we have here used the simplified notations

$$\mu^\lambda := \mu + \lambda\eta \quad (4.15) \quad \{\text{mul}\}$$

and by X^{μ^λ} we mean the corresponding solution

$$X^{\mu^\lambda}(t) = x + \int_0^t \int_{\mathbb{R}_0} \gamma(s, X^{\mu^\lambda}(s), \mu^\lambda(s), \zeta) \tilde{N}(ds, d\zeta); \quad t \in [0, T],$$

when assuming that $b = \sigma = 0$, and because u is fixed we can omit it. Then we get

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \left(\frac{X^{\mu^\lambda}(s) - X^\mu(s)}{\lambda} - Z(s) \right)^2 ds \right] \\ &= \mathbb{E} \left[\int_0^T \left(\int_{\mathbb{R}_0} \left\{ \frac{\gamma(s, X^{\mu^\lambda}(s), \mu^\lambda(s), \zeta) - \gamma(s, X^\mu(s), \mu(s), \zeta)}{\lambda} - \frac{\partial \gamma}{\partial x}(s, \zeta) Z(s) \right\} \tilde{N}(ds, d\zeta) \right)^2 \right] \\ &\leq \mathbb{E} \left[\int_0^T \int_{\mathbb{R}_0} \left(\frac{\gamma(s, X^{\mu^\lambda}(s), \mu^\lambda(s), \zeta) - \gamma(s, X^\mu(s), \mu(s), \zeta)}{\lambda} - \frac{\partial \gamma}{\partial x}(s, \zeta) Z(s) \right)^2 \nu(d\zeta) ds \right]. \end{aligned}$$

This goes to 0 when λ goes to 0, by Kunita's inequality, the bounded convergence theorem and our assumption on γ . □

Theorem 4.5 (Necessary nonzero-sum maximum principle) *Let $(\hat{\mu}, \hat{u}) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ with corresponding solutions \hat{X} and $(\hat{p}_i, \hat{q}_i, \hat{r}_i)$ of the forward and backward stochastic differential equations (3.1) and (4.3) – (4.4), with the corresponding derivative process \hat{Z} given by (4.13). Then the following (i) and (ii) are equivalent:*

(i) *For all $\mu, \eta \in \mathbb{M}_{\mathbb{G}}$ and for all $u, \pi \in \mathcal{A}_{\mathbb{G}}$*

$$\frac{d}{d\lambda} J_1(\mu + \lambda\eta, u) \Big|_{\lambda=0} = \frac{d}{ds} J_2(\mu, u + s\pi) \Big|_{s=0} = 0,$$

(ii)

$$\mathbb{E} \left[\frac{\partial H_1}{\partial \mu}(t) \Big| \mathcal{G}_t^{(1)} \right] = \mathbb{E} \left[\frac{\partial H_2}{\partial u}(t) \Big| \mathcal{G}_t^{(2)} \right] = 0.$$

Proof. Before starting the proof, let us first introduce some notations: Note that

$$\nabla_m \langle p_1^1(t), \frac{d}{dt} m \rangle = \langle p_1^1(t), \frac{d}{dt}(\cdot) \rangle,$$

and hence

$$\langle \nabla_m \langle p_1^1(t), \frac{d}{dt} m \rangle, DM(t) \rangle = \langle p_1^1(t), \frac{d}{dt} DM(t) \rangle = \langle p_1^1(t), DM'(t) \rangle = p_1^1(t) DM'(t).$$

Also, note that

$$\begin{aligned} DM(t) &= \frac{d}{d\lambda} \mathcal{L}(X^{\mu+\lambda\eta}) \Big|_{\lambda=0} = \mathcal{L}(Z(t)) \\ \mathcal{L}(Z(t)) &= dDM(t) = DM'(t) dt. \end{aligned}$$

Assume that (i) holds and using the definition of J_1 (4.1),

$$\begin{aligned} 0 &= \frac{d}{d\lambda} J_1(\mu + \lambda\eta, u) \Big|_{\lambda=0} \\ &= \mathbb{E} \left[\int_0^T \left\{ \frac{\partial \ell_1}{\partial x}(t) Z(t) + \nabla_m \ell_1(t) DM(t) + \frac{\partial \ell_1}{\partial \mu}(t) \eta(t) \right\} dt \right. \\ &\quad \left. + \frac{\partial g_1}{\partial x}(T) Z(T) + \nabla_m g_1(T) DM(T) \right]. \end{aligned} \tag{4.16}$$

Hence, by the definition of H_1 (4.2), we have

$$\begin{aligned} 0 &= \frac{d}{d\lambda} J_1(\mu + \lambda\eta, u) \Big|_{\lambda=0} \\ &= \mathbb{E} \left[\int_0^T \left\{ \frac{\partial H_1}{\partial x}(t) - p_1^0(t) \frac{\partial b}{\partial x}(t) - q_1^0(t) \frac{\partial \sigma}{\partial x}(t) - \int_{\mathbb{R}_0} r_1^0(t, \zeta) \frac{\partial \gamma}{\partial x}(t, \zeta) \nu(d\zeta) Z(t) \right\} dt \right. \\ &\quad + \int_0^T \nabla_m H_1(t) DM(t) dt - \int_0^T p_1^1(t) DM'(t) dt + \int_0^T \frac{\partial H_1}{\partial \mu}(t) - p_1^0(t) \frac{\partial b}{\partial \mu}(t) \\ &\quad \left. - q_1^0(t) \frac{\partial \sigma}{\partial \mu}(t) - \int_{\mathbb{R}_0} r_1^0(t, \zeta) \frac{\partial \gamma}{\partial \mu}(t, \zeta) \nu(d\zeta) \right\} \eta(t) dt + p_1^0(T) Z(T) + p_1^1(T) DM(T) \Big]. \end{aligned} \tag{4.17} \quad \{\mathbf{dj}\}$$

Applying now the Itô formula to both $p_1^0 Z$ and $p_1^1 DM$, we get

$$\begin{aligned}
& \mathbb{E}[p_1^0(T)Z(T) + p_1^1(T)DM(T)] \\
&= \mathbb{E}[\int_0^T p_1^0(t)dZ(t) + \int_0^T Z(t)dp_1^0(t) + \int_0^T q_1^0(t)(\frac{\partial \sigma}{\partial x}(t)Z(t) + \frac{\partial \sigma}{\partial \mu}(t)\eta(t))dt \\
&+ \int_0^T \int_{\mathbb{R}_0} r_1^0(t, \zeta)(\frac{\partial \gamma}{\partial x}(t, \zeta)Z(t) + \frac{\partial \gamma}{\partial \mu}(t, \zeta)\eta(t))\nu(d\zeta)dt] \\
&+ \mathbb{E}\left[\int_0^T p_1^1(t)DM'(t)dt + \int_0^T DM(t)dp_1^1(t)\right] \\
&= \mathbb{E}\left[\int_0^T p_1^0(t)(\frac{\partial b}{\partial x}(t)Z(t) + \frac{\partial b}{\partial \mu}(t)\eta(t))dt - \int_0^T \frac{\partial H_1}{\partial x}(t)Z(t)dt \right. \\
&+ \int_0^T q_1^0(t)(\frac{\partial \sigma}{\partial x}(t)Z(t) + \frac{\partial \sigma}{\partial \mu}(t)\eta(t))dt \\
&+ \int_0^T \int_{\mathbb{R}_0} r_1^0(t, \zeta)(\frac{\partial \gamma}{\partial x}(t, \zeta)Z(t) + \frac{\partial \gamma}{\partial \mu}(t, \zeta)\eta(t))\nu(d\zeta)dt \\
&\left. + \int_0^T p_1^1(t)DM'(t)dt - \int_0^T \nabla_m H_1(t)DM(t)dt\right]. \tag{4.18}
\end{aligned}$$

Combining the above, we get

$$0 = \mathbb{E}\left[\int_0^T \frac{\partial H_1}{\partial \mu}(t)\eta(t)dt\right] = \mathbb{E}\left[\int_s^T \frac{\partial H_1}{\partial \mu}(t)\alpha_1 dt\right]; s \geq t_0.$$

Differentiating with respect to s we obtain

$$\begin{aligned}
0 &= \mathbb{E}\left[\frac{\partial H_1}{\partial \mu}(s)\alpha_1\right] \\
&= \mathbb{E}\left[\frac{\partial H_1}{\partial \mu}(t_0)\middle|\mathcal{G}_{t_0}^{(1)}\right],
\end{aligned}$$

because this holds for all α_1 and all $s \geq t_0$.

This argument can be reversed, to prove that (ii) \implies (i). We omit the details.

In the same manner, we can get the equivalence between

$$\frac{d}{ds}J_2(\mu, u + s\pi)\big|_{s=0} = 0$$

and

$$\mathbb{E}\left[\frac{\partial H_2}{\partial u}(t)\middle|\mathcal{G}_t^{(2)}\right] = 0.$$

□

In the next section we will consider the zero-sum case, and find conditions for a saddle point of such games.

5 Zero-sum game

In this section, we proceed to study the maximum principle for the zero-sum game case. Let us then define the performance functional as

$$J(\mu, u) = \mathbb{E}[g(X(T), M(T))] + \int_0^T \ell(s, X(s), M(s), \mu(s), u(s)) ds],$$

where the state $X(t)$ is the solution of a SDE (3.1).

The functions

$$\ell(s, x, m, \mu, u) = \ell(s, x, m, \mu, u, \omega) : [0, T] \times \mathbb{R} \times \mathcal{M} \times \mathcal{M} \times \mathcal{U} \times \Omega \rightarrow \mathbb{R}$$

and

$$g(x, m) = g(x, m, \omega) : \mathbb{R} \times \mathcal{M} \times \Omega \rightarrow \mathbb{R}$$

are supposed to satisfy the following conditions:

(a) ℓ and g are continuously differentiable w.r.t x, u and admits Fréchet derivatives w.r.t m and μ .

(b) Moreover, the function

$$\mathbb{R} \times \mathcal{M} \ni (x, m) \mapsto g(x, m)$$

is required to be affine a.s. \mathbb{P} .

We consider the stochastic zero-sum game to find (μ^*, u^*) such that

$$\sup_{u \in \mathcal{A}_G} \inf_{\mu \in \mathbb{M}_G} J(\mu, u) = \inf_{\mu \in \mathbb{M}_G} \sup_{u \in \mathcal{A}_G} J(\mu, u) = J(\mu^*, u^*).$$

We call (μ^*, u^*) a *saddle point* for $J(\mu, u)$.

In this case, let the Hamiltonian $H : [0, T] \times \mathbb{R} \times C_a(\mathcal{M}_0) \times \mathcal{M} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \mathcal{K} \rightarrow \mathbb{R}$ be given by

$$\begin{aligned} H(t, x, m, \mu, u, p, q, r(\cdot), p^1) &= \ell(t, x, m, \mu, u) + p^0 b(t, x, \mu, u) \\ &+ q^0 \sigma(t, x, \mu, u) + \int_{\mathbb{R}_0} r^0(\zeta) \gamma(t, x, \mu, u, \zeta) \nu(d\zeta) + p^1 m'. \end{aligned}$$

We assume the following:

(c) H is continuously differentiable w.r.t x, u and admits Fréchet derivatives w.r.t m and μ .

(d) The Hamiltonian function

$$\mathbb{R} \times C_a(\mathcal{M}_0) \times \mathcal{M} \times \mathcal{U} \ni (x, m, \mu, u) \mapsto H(t, x, m, \mu, u, p, q, r(\cdot), p^1)$$

is *convex* w.r.t. (x, m, μ) and *concave* w.r.t. (x, m, u) \mathbb{P} .a.s and for each $t \in [0, T]$, $p, q, r(\cdot)$ and p^1 .

For $u \in \mathcal{A}_G, \mu \in \mathbb{M}_G$ with corresponding solution $X = X^{\mu, u}$, define $p = p^{\mu, u}, q = q^{\mu, u}$ and $r = r^{\mu, u}$ by the adjoint equations:

The real-BSDE in the unknown $(p^0, q^0, r^0) \in \mathcal{S}^2 \times L^2 \times L^2_{\nu}$ has the following form

$$\begin{cases} dp^0(t) &= -\frac{\partial H}{\partial x}(t) dt + q^0(t)dB(t) + \int_{\mathbb{R}_0} r^0(t, \zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ p^0(T) &= \frac{\partial g}{\partial x}(X(T), M(T)), \end{cases} \quad (5.1) \quad \{\text{pro}\}$$

and the operator-valued BSDE for the unknown $(p^1, q^1, r^1) \in \mathcal{S}^2 \times L^2 \times L^2_\nu$ is given by

$$\begin{cases} dp^1(t) &= -\nabla_m H(t)dt + q^1(t)dB(t) + \int_{\mathbb{R}_0} r^1(t, \zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ p^1(T) &= \nabla_m g(X(T), M(T)). \end{cases} \quad (5.2) \quad \{\text{pro1}\}$$

Theorem 5.1 (Sufficient zero-sum maximum principle) *Let $(\hat{\mu}, \hat{u}) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ with corresponding solutions $\hat{X} \in \mathcal{S}^2$ and $(p^0, q^0, r^0), (p^1, q^1, r^1) \in \mathcal{S}^2 \times L^2 \times L^2_\nu$ of the forward and backward stochastic differential equations (3.1), (5.1) – (5.2). Assume the following:*

•

$$\mathbb{E} \left[\hat{H}(t) \middle| \mathcal{G}_t^{(1)} \right] = \operatorname{ess\,sup}_{\mu \in \mathbb{M}_{\mathbb{G}}} \mathbb{E} \left[\check{H}(t) \middle| \mathcal{G}_t^{(1)} \right],$$

•

$$\mathbb{E} \left[\bar{H}(t) \middle| \mathcal{G}_t^{(2)} \right] = \operatorname{ess\,sup}_{u \in \mathcal{A}_{\mathbb{G}}} \mathbb{E} \left[\check{\check{H}}(t) \middle| \mathcal{G}_t^{(2)} \right],$$

\mathbb{P} .a.s and for all $t \in [0, T]$, and that assumptions (a)-(d) hold.

Then $(\hat{\mu}, \hat{u})$ is a saddle point for $J(\mu, u)$.

This result will be applied in the next section.

Theorem 5.2 (Necessary zero-sum maximum principle) *Let $(\hat{\mu}, \hat{u}) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ with corresponding solutions \hat{X} and $(\hat{p}, \hat{q}, \hat{r})$ of the forward and the backward stochastic differential equations (3.1) and (5.1) – (5.2), respectively, with corresponding derivative process \hat{Z} given by (4.13). Then we have equivalence between*

$$\left. \frac{d}{d\lambda} J(\mu + \lambda\eta, u) \right|_{\lambda=0} = \left. \frac{d}{ds} J(\mu, u + s\pi) \right|_{s=0} = 0,$$

and

$$\mathbb{E} \left[\frac{\partial H}{\partial \mu}(t) \middle| \mathcal{G}_t^{(1)} \right] = \mathbb{E} \left[\frac{\partial H}{\partial u}(t) \middle| \mathcal{G}_t^{(2)} \right] = 0.$$

Proof. The same proof of both the sufficient and the necessary maximum principles for the nonzero-sum games works for the zero-sum case. \square

6 Optimal consumption of a mean-field cash flow under uncertainty

Consider a net cash flow $X^{\mu, \rho} = X$ modeled by

$$\begin{cases} dX(t) = [\mu(t)(V) - \rho(t)] X(t)dt + \sigma(t) X(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t, \zeta) X(t) \tilde{N}(dt, d\zeta); t \in [0, T], \\ X(0) = x > 0, \end{cases}$$

where $\rho(t) \geq 0$ is our *relative consumption rate* at time t , assumed to be a càdlàg, $\mathcal{G}_t^{(2)}$ -adapted process. Here V is a given Borel subset of \mathbb{R} . The value of $\mu(t)$ on V models the relative growth rate of the cash flow. The relative consumption rate $\rho(t)$ is our control process. We assume that $\int_0^T \rho(t)dt < \infty$ a.s. This implies that $X(t) > 0$ for all t , a.s. However, the measure-valued process $\mu(t)$ represents a kind of scenario uncertainty, and we want to maximise the total expected utility of the relative consumption rate ρ in the worst possible scenario μ . We penalize $\mu(\cdot)$ for being far away from the law process $\mathcal{L}(X(\cdot))$, in the sense that we introduce a quadratic cost rate $[(\mu(t) - \mathcal{L}(X(t)))(V)]^2$ in the performance functional. Hence we consider the zero-sum game

$$\sup_{\rho} \inf_{\mu} \mathbb{E}[\int_0^T \{\log(\rho(t)X(t)) + [(\mu(t) - \mathcal{L}(X(t)))(V)]^2\}dt + \theta \log(X(T))],$$

where $\theta = \theta(\omega) > 0$ is a given bounded \mathcal{F}_T -measurable random variable, expressing the importance of the terminal value $X(T)$. *Here we have chosen a logarithmic utility because it is a central choice, and in many cases, as here, this leads to a nice explicit solution of the corresponding control problem.*

The Hamiltonian for this zero-sum game takes the form

$$\begin{aligned} H(t) = & \log(\rho x) + (\mu(V)x - m(V))^2 + p^0[\mu(V)x - \rho x] + q^0\sigma(t)x \\ & + \int_{\mathbb{R}_0} r^0(\zeta)\gamma(t, \zeta)x\nu(d\zeta) + p^1m', \end{aligned}$$

and the adjoint processes $(p^0, q^0, r^0), (p^1, q^1, r^1) \in \mathcal{S}^2 \times L^2 \times L^2_\nu$ are given by the BSDEs

•

$$\begin{cases} dp^0(t) = - \left[\frac{1}{X(t)} + p^0(t)[\mu(t)(V) - \rho(t)] + q^0(t)\sigma(t) + \int_{\mathbb{R}_0} r^0(t, \zeta)\gamma(t, \zeta)\nu(d\zeta) \right] dt \\ \quad + q^0(t)dB(t) + \int_{\mathbb{R}_0} r^0(t, \zeta)\tilde{N}(dt, d\zeta); t \in [0, T], \\ p^0(T) = \frac{\theta}{X(T)}, \end{cases}$$

•

$$\begin{cases} dp^1(t) = -2[\hat{\mu}(t)(V) - \hat{M}(t)(V)]\chi_V(\cdot)dt + q^1(t)dB(t) + \int_{\mathbb{R}_0} r^1(t, \zeta)\tilde{N}(dt, d\zeta); t \in [0, T], \\ p^1(T) = 0, \end{cases}$$

where $\chi_V(\cdot)$ is the operator which evaluates a given measure at V , i.e. $\langle \chi_V, \lambda \rangle = \lambda(V)$ for all $\lambda \in \mathcal{M}_0$. The first order condition for the optimal consumption rate $\hat{\rho}$ is

$$\mathbb{E} \left[\frac{1}{\hat{\rho}(t)} - \hat{p}^0(t) \hat{X}(t) \middle| \mathcal{G}_t^{(2)} \right] = 0.$$

Since $\hat{\rho}(t)$ is $\mathcal{G}_t^{(2)}$ -adapted, we have

$$\hat{\rho}(t) = \frac{1}{\mathbb{E}[\hat{p}^0(t) \hat{X}(t) | \mathcal{G}_t^{(2)}]}. \quad (6.1)$$

Now we use the minimum condition w.r.t μ at $\mu = \hat{\mu}$ and get

$$\mathbb{E} \left[2[\hat{\mu}(t)(V) - \hat{M}(t)(V)]\lambda(V) + \hat{p}^0(t) \hat{X}(t) \lambda(V) \middle| \mathcal{G}_t^{(1)} \right] = 0, \text{ for all } \lambda \in \mathcal{M}_0.$$

Using that $\hat{\mu}(t)$ is $\mathcal{G}_t^{(1)}$ -adapted, we obtain

$$\hat{\mu}(t)(V) = \mathbb{E} \left[\mathcal{L}(\hat{X}(t))(V) - \frac{1}{2} \hat{p}^0(t) \hat{X}(t) \middle| \mathcal{G}_t^{(1)} \right]. \quad (6.2)$$

It remains to find $\hat{p}^0(t) \hat{X}(t)$: We have by applying Itô formula to $P(t) := \hat{p}^0(t) \hat{X}(t)$

$$\begin{aligned} dP(t) &= \hat{p}^0(t) d\hat{X}(t) + \hat{X}(t) d\hat{p}^0(t) + d[\hat{p}^0, \hat{X}]_t \\ &= \hat{p}^0(t) \left([\hat{\mu}(t)(V) - \rho(t)] \hat{X}(t) \right) dt + \hat{\sigma}(t) \hat{X}(t) dB(t) + \int_{\mathbb{R}_0} \hat{\gamma}(t, \zeta) \hat{X}(t) \tilde{N}(dt, d\zeta) \\ &\quad + \hat{X}(t) \left[-\frac{1}{\hat{X}(t)} - \hat{p}^0(t) [\hat{\mu}(t)(V) - \rho(t)] - \hat{q}^0(t) \sigma(t) - \int_{\mathbb{R}_0} \hat{r}^0(t, \zeta) \hat{\gamma}(t, \zeta) \nu(d\zeta) \right] dt \\ &\quad + \hat{q}^0(t) \hat{X}(t) dB(t) + \int_{\mathbb{R}_0} \hat{r}^0(t, \zeta) \hat{X}(t) \tilde{N}(dt, d\zeta) + \hat{q}^0(t) \hat{\sigma}(t) \hat{X}(t) dt \\ &\quad + \int_{\mathbb{R}_0} \hat{r}^0(t, \zeta) \hat{\gamma}(t, \zeta) \hat{X}(t) N(dt, d\zeta). \end{aligned} \quad (6.3) \quad \{\text{na1}\}$$

By definition

$$\begin{aligned} \int_{\mathbb{R}_0} \hat{r}^0(t, \zeta) \hat{\gamma}(t, \zeta) \hat{X}(t) \tilde{N}(dt, d\zeta) &= \int_{\mathbb{R}_0} \hat{r}^0(t, \zeta) \hat{\gamma}(t, \zeta) \hat{X}(t) N(dt, d\zeta) \\ &\quad - \int_{\mathbb{R}_0} \hat{r}^0(t, \zeta) \hat{\gamma}(t, \zeta) \hat{X}(t) \nu(d\zeta) dt. \end{aligned} \quad (6.4) \quad \{\text{na2}\}$$

Substituting (6.4) in (6.3) yields

$$\begin{aligned} dP(t) &= -dt + \left[P(t) \hat{\sigma}(t) + \hat{q}^0(t) \hat{X}(t) \right] dB(t) \\ &\quad + \int_{\mathbb{R}_0} \left[P(t) \hat{\gamma}(t, \zeta) + \hat{r}^0(t, \zeta) \hat{X}(t) (1 + \hat{\gamma}(t, \zeta)) \right] \tilde{N}(dt, d\zeta). \end{aligned}$$

Hence, if we put

$$\begin{aligned} P(t) &:= \hat{p}^0(t) \hat{X}(t), \\ Q(t) &:= P(t) \hat{\sigma}(t) + \hat{X}(t) \hat{q}^0(t), \\ R(t, \zeta) &:= P(t) \hat{\gamma}(t, \zeta) + \hat{r}^0(t, \zeta) \hat{X}(t) (1 + \hat{\gamma}(t, \zeta)). \end{aligned}$$

with $(P, Q, R) \in \mathcal{S}^2 \times L^2 \times L^2_\nu$ satisfies the BSDE

$$\begin{cases} dP(t) &= -dt + Q(t)dB(t) + \int_{\mathbb{R}_0} R(t, \zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ P(T) &= \theta. \end{cases}$$

Solving this BSDE as in (4.6), we find that

$$\begin{aligned} P(t) &= \mathbb{E} \left[\theta + \int_t^T ds \middle| \mathcal{G}_t \right] \\ &= \mathbb{E} [\theta | \mathcal{G}_t] + T - t. \end{aligned}$$

Hence we have proved the following:

Theorem 6.1 *The optimal consumption rate $\hat{\rho}(t)$ and the optimal model uncertainty law $\hat{\mu}(t)$ are given respectively in feed-back form by*

$$\begin{aligned} \hat{\rho}(t) &= \frac{1}{T - t + \mathbb{E} [\theta | \mathcal{G}_t^{(2)}]}, \\ \hat{\mu}(t)(V) &= \mathcal{L}(\hat{X}(t))(V) + T - t - \frac{1}{2} \mathbb{E} [\theta | \mathcal{G}_t^{(1)}]. \end{aligned}$$

Remark 6.2 Theorem 6.1 shows that the optimal scenario $\hat{\mu}(t)$ differs from the corresponding state process law $\mathcal{L}(\hat{X}(t))$ at V by

$$(\hat{\mu}(t) - \mathcal{L}(\hat{X}(t)))(V) = T - t - \frac{1}{2} \mathbb{E} [\theta | \mathcal{G}_t^{(1)}].$$

In particular if $\mathcal{G}_t^{(1)} = \mathcal{G}_t^{(2)}$ then

$$(\hat{\mu}(t) - \mathcal{L}(\hat{X}(t)))(V) = \frac{1}{\hat{\rho}(t)}.$$

7 Appendix

We want to prove that $J_2(\hat{\mu}, u) \leq J_2(\hat{\mu}, \hat{u})$. Using definition (4.1) gives for fixed $\hat{\mu} \in \mathbb{M}_{\mathbb{G}}$ and an arbitrary $u \in \mathcal{A}_{\mathbb{G}}$

$$J_2(\hat{\mu}, u) - J_2(\hat{\mu}, \hat{u}) = j_1 + j_2, \tag{7.1} \quad \{\text{J2}\}$$

where

$$\begin{aligned} j_1 &= \mathbb{E} \left[\int_0^T \left\{ \check{\ell}_2(t) - \bar{\ell}_2(t) \right\} dt \right], \\ j_2 &= \mathbb{E} \left[\check{g}_2(X(T), M(T)) - \bar{g}_2(\hat{X}(T), \hat{M}(T)) \right]. \end{aligned}$$

Applying the definition of the Hamiltonian (4.2) we have

$$\begin{aligned} j_1 &= \mathbb{E} \left[\int_0^T \left\{ \check{H}_2(t) - \check{H}_2(t) - \hat{p}_2^0(t) \tilde{b}(t) - \hat{q}_2^0(t) \tilde{\sigma}(t) \right. \right. \\ &\quad \left. \left. - \int_{\mathbb{R}_0} \hat{r}_2^0(t, \zeta) \tilde{\gamma}(t, \zeta) \nu(d\zeta) - \hat{p}_2^1(t) \tilde{M}'(t) \right\} dt \right], \end{aligned} \tag{7.2} \quad \{\text{j1}\}$$

where $\tilde{b}(t) = \check{b}(t) - \bar{b}(t)$. etc., and

$$\tilde{M}'(t) = \frac{d\tilde{M}(t)}{dt} = \frac{d}{dt}\mathcal{L}(\tilde{X}(t)).$$

Concavity of g_2 and the definition of the terminal value of the BSDEs (4.3) and (4.4) shows that

$$\begin{aligned} j_2 &\leq \mathbb{E}[\frac{\partial g_2}{\partial x}(T)\tilde{X}(T) + \nabla_m g_2(T)\tilde{M}(t)] \\ &= \mathbb{E}[\hat{p}_2^0(T)\tilde{X}(T) + \hat{p}_2^1(T)\tilde{M}(t)]. \end{aligned} \tag{7.3} \quad \{j_2\}$$

Applying the Itô formula to $\hat{p}_2^0\tilde{X}$ and $\hat{p}_2^1\tilde{M}$, we get

$$\begin{aligned} j_2 &\leq \mathbb{E}[\hat{p}_2^0(T)\tilde{X}(T) + \hat{p}_2^1(T)\tilde{M}(T)] \\ &= \mathbb{E}\left[\int_0^T \hat{p}_2^0(t)d\tilde{X}(t) + \int_0^T \tilde{X}(t)d\hat{p}_2^0(t) + \int_0^T \hat{q}_2^0(t)\tilde{\sigma}(t)dt + \int_0^T \int_{\mathbb{R}_0} \hat{r}_2^0(t, \zeta)\tilde{\gamma}(t, \zeta)\nu(d\zeta)dt\right] \\ &\quad + \mathbb{E}\left[\int_0^T \hat{p}_2^1(t)d\tilde{M}(t) + \int_0^T \tilde{M}(t)d\hat{p}_2^1(t)\right] \\ &= \mathbb{E}\left[\int_0^T \hat{p}_2^0(t)\tilde{b}(t)dt - \int_0^T \frac{\partial \bar{H}_2}{\partial x}(t)\tilde{X}(t)dt + \int_0^T \hat{q}_2^0(t)\tilde{\sigma}(t)dt\right. \\ &\quad \left.+ \int_0^T \int_{\mathbb{R}_0} \hat{r}_2^0(t, \zeta)\tilde{\gamma}(t, \zeta)\nu(d\zeta)dt + \int_0^T \hat{p}_2^1(t)\tilde{M}'(t)dt - \int_0^T \nabla_m \bar{H}_2(t)\tilde{M}(t)dt\right], \end{aligned}$$

where we have used that the $dB(t)$ and $\tilde{N}(dt, d\zeta)$ integrals have mean zero. Substituting (7.2) and (7.3) into (7.1), we obtain

$$J_2(\hat{\mu}, u) - J_2(\hat{\mu}, \hat{u}) \leq \mathbb{E}\left[\int_0^T \{\check{H}_2(t) - \bar{H}_2(t) - \frac{\partial \bar{H}_2}{\partial x}(t)\tilde{X}(t) - \nabla_m \bar{H}_2\tilde{M}(t)\}dt\right].$$

Since H_2 is concave and the process u is $\mathcal{G}_t^{(2)}$ -adapted, we have

$$\begin{aligned} J_2(\hat{\mu}, u) - J_2(\hat{\mu}, \hat{u}) &\leq \mathbb{E}\left[\int_0^T \frac{\partial \bar{H}_2}{\partial u}(t)(u(t) - \hat{u}(t))dt\right] \\ &= \mathbb{E}\left[\int_0^T \mathbb{E}\left[\frac{\partial \bar{H}_2}{\partial u}(t) \middle| \mathcal{G}_t^{(2)}\right](u(t) - \hat{u}(t))dt\right] \\ &\leq 0, \end{aligned}$$

because \bar{H}_2 has a maximum at \hat{u} . □

Acknowledgment

We are grateful to Youssef Ouknine and his group in Marrakech for helpful comments.

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